ACC Precalculus
Determinants and Cramer's Rule

Name ______________________ .
Date _____________ Block _____ .

Determinant of a 2 X 2 Matrix

Associated with each square matrix (nxn) is a real number called its determinant.
The **determinant** of a matrix A is denoted by $\det A$ or by $|A|$.

Determinant of a 2×2 Matrix

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

The determinant of a 2×2 matrix is the difference of the products of the elements on the diagonals.

Evaluate ~~each determinant~~.

1. $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix}$ $5-6=-1$

2. $\begin{vmatrix} -4 & 3 \\ -5 & 2 \end{vmatrix}$ $(-4\cdot 2)-(-5\cdot 3)$
$-8+15=7$

3. Solve for x given $\begin{vmatrix} x & x \\ -5 & x \end{vmatrix} = 24$. $x^2-(-5x)$

$9+15$ ✓

$x^2+5x=24$
$x^2+5x-24=0$
$(x+8)(x-3)=0$
$\boxed{x=-8,3}$

Determinant of a 3 X 3 Matrix

Steps for finding the determinant of a $3x3$ Matrix

- Repeat the first two columns to the right of the determinant.
- Subtract the sum of the **products in red** from the sum of **the products in blue**.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}\begin{matrix} a & b \\ d & e \\ g & h \end{matrix} = (aei+bfg+cdh)-(gec+hfa+idb)$$

4. Given $A = \begin{bmatrix} 2 & -1 & 3 \\ -2 & 0 & 1 \\ 1 & 2 & 4 \end{bmatrix}$, find det($A$).

$$\begin{vmatrix} 2 & -1 & 3 \\ -2 & 0 & 1 \\ 1 & 2 & 4 \end{vmatrix}\begin{matrix} 2 & -1 \\ -2 & 0 \\ 1 & 2 \end{matrix}$$

$(2\cdot 0\cdot 4 + -1\cdot 1\cdot 1 + 3\cdot -2\cdot 2) -$
$(1\cdot 0\cdot 3 + 2\cdot 1\cdot 2 + 4\cdot -2\cdot -1)$

$(0-1-12)-(0+4+8)$
$-13-12=-25$

5. Evaluate $\begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 5 \\ 3 & 6 & 7 \end{vmatrix}$ $\begin{matrix} 1 & 3 \\ 2 & 1 \\ 3 & 6 \end{matrix}$ $= (7+45+48) - (12+30+42)$

$100 - 84$

16

Special Determinants

Triangular Matrices – When the numbers below or above a diagonal in a square matrix are all zeros, the determinant can be found by multiplying the numbers that lie on the diagonal.

6. $\begin{vmatrix} 2 & 5 & -9 \\ 0 & 3 & 8 \\ 0 & 0 & -3 \end{vmatrix}$ $\begin{matrix} 2 & 5 \\ 0 & 3 \\ 0 & 0 \end{matrix}$

$(-18+0+0) - (0+0+0)$

$= -18$

7. $\begin{vmatrix} 0 & 0 & 0 & -3 \\ 0 & 0 & 5 & 8 \\ 0 & 2 & 7 & 3 \\ -1 & 4 & 6 & 3 \end{vmatrix}$

$= 30$

$-1 \cdot 2 \cdot 5 \cdot -3$

8. $\begin{vmatrix} 6 & 9 & 7 & 2 \\ 3 & 2 & 7 & 0 \\ 0 & 5 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{vmatrix}$

$3 \cdot 5 \cdot 7 \cdot 2$

$\boxed{210}$

Practice: **You will need to show work clearly on a separate sheet of paper.**

Find the value of each given determinant.

1) $\begin{vmatrix} 3 & 6 \\ 5 & 3 \end{vmatrix} = -21$

2) $\begin{vmatrix} 6 & -2 \\ 2 & 1 \end{vmatrix} = 10$

3) $\begin{vmatrix} 4 & -2 \\ -8 & 4 \end{vmatrix} = 0$

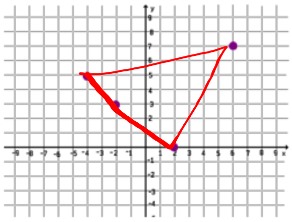
4) $\begin{vmatrix} 4 & 3 & 1 \\ 0 & 2 & 3 \\ 4 & 2 & 5 \end{vmatrix} = 44$

5) $\begin{vmatrix} 5 & 2 & -1 \\ 3 & -2 & 5 \\ 1 & 1 & 3 \end{vmatrix} = -68$

6) $\begin{vmatrix} 0 & 3 & 2 \\ -4 & -1 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$

Determinant Application: Finding the area of a polygon given its vertices.
The vertices must be ***in order***. Even though we are not actually finding the determinant we can still get the area by crossing out diagonals.

Example: Find the area of the quadrilateral with vertices (-2, 3), (-4, 5), (6, 7), and (2, 0). ***These points must be written in order***. You can start with any of the points, but the following points must be written consecutively. Graph to make sure they are in order.



In the case of 4 vertices:

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_1 \\ y_1 & y_2 & y_3 & y_4 & y_1 \end{vmatrix}$$ *write 1st point again at the end*

Find the sum of the diagonals going down to the right and subtract the sum of the diagonals going up to the right. Remember that area is always positive.

9. Our example: (-4, 5), (-2, 3), (2, 0), (6, 7) ~~I have reordered according to x-values.~~

$$\text{Area} = \pm\frac{1}{2}\begin{vmatrix} -4 & -2 & 2 & 6 & -4 \\ 5 & 3 & 0 & 7 & 5 \end{vmatrix}$$

$$\pm\frac{1}{2}\left[(-12+0+14+30)-(-10+6+0-28)\right]$$

$$\frac{1}{2}\left[32-(-32)\right]$$

$$\frac{1}{2}(64) = \underline{32} \text{ sq. units}$$

Try these: Find the area of the triangles given 3 points.

10. (-2, 1) (4, 7) (9, 8)

$$\pm\frac{1}{2}\begin{vmatrix} -2 & 4 & 9 & -2 \\ 1 & 7 & 8 & 1 \end{vmatrix}$$

$$\pm\frac{1}{2}\left[(-14+32+9)-(4+63-16)\right]$$

$$\pm\frac{1}{2}(27-51) = -\frac{1}{2}(-24) = 12 \text{ sq units}$$

11. (-2, 3) (4, 7) (0, -1)

$$\pm\frac{1}{2}\begin{vmatrix} -2 & 3 & 1 & -2 & 3 \\ 4 & 7 & 1 & 4 & 7 \\ 0 & -1 & 1 & 0 & -1 \end{vmatrix}$$

$$(-14+0-4)-(0+2+12)$$

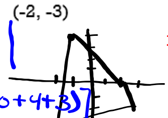
$$\pm\frac{1}{2}(-18-14)$$

$$-\frac{1}{2}(-32) = 16 \text{ units}^2$$

Find the area of the polygon given the vertices; make sure you put the points in order.

12. (-1, 4) (2, 0) (3, -2) (-2, -3)

$$\pm\frac{1}{2}\begin{vmatrix} -1 & 2 & 3 & -2 & -1 \\ 4 & 0 & -2 & -3 & 4 \end{vmatrix}$$



$$\pm\frac{1}{2}\left[(0-4-9-8)-(8+0+4+3)\right]$$

$$-\frac{1}{2}(-21-15) = 18 \text{ units}^2$$

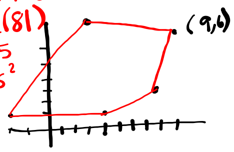
13. (4, 1) (9, 6) (3, 7) (8, 3) (-2, 1)

$$\pm\frac{1}{2}\begin{vmatrix} -2 & 4 & 8 & 9 & 3 & -2 \\ 1 & 1 & 3 & 6 & 7 & 1 \end{vmatrix}$$

$$\pm\frac{1}{2}\left[(-2+12+48+63+3)-(4+8+27+18-14)\right]$$

$$\frac{1}{2}(124-43) = -\frac{1}{2}(81)$$

40.5 units^2



14. The Bermuda Triangle is a large triangular region in the Atlantic Ocean. Many ships and airplanes have been lost in this region. The triangle is formed by imaginary lines connecting Bermuda, Puerto Rico, and Miami. Use a determinant to estimate the area of the Bermuda Triangle.

$$\pm\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 938 & 454 & 1 \\ 900 & -518 & 1 \end{vmatrix}$$

*I used the cofactor method (video) to find the determinant.

Det

$$0\begin{vmatrix} 454 & 1 \\ -518 & 1 \end{vmatrix} - 0\begin{vmatrix} 938 & 1 \\ 900 & 1 \end{vmatrix} + 1\begin{vmatrix} 938 & 454 \\ 900 & -518 \end{vmatrix} =$$

ad − bc

$$-485{,}884 - 408{,}600$$

$$= -894{,}484$$

$$\text{Area} = -\frac{1}{2}(-894{,}484)$$

$$= \boxed{447{,}242 \text{ sq miles}}$$

Solve a 2 X 2 Sytem Using Cramer's Rule

You can use determinants to solve a system of linear equations. This method, called Cramer's rule, uses the coefficient matrix of the linear system.

Linear System

$ax + by = e$
$cx + dy = f$

Coefficient Matrix

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Cramer's Rule for a 2x2 System

Let A be the coefficient matrix of this linear system:

$$ax + by = e$$
$$cx + dy = f$$

$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

If $\det A \neq 0$, then the system has exactly one solution. The solution is:

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\det A} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\det A}$$

15. Use Cramer's Rule to solve this system: $8x + 5y = 2$; $2x - 4y = -10$

$|A| = \begin{vmatrix} 8 & 5 \\ 2 & -4 \end{vmatrix}$

$-32 - 10 = -42$

$x = \dfrac{\begin{vmatrix} 2 & 5 \\ -10 & -4 \end{vmatrix}}{-42} = \dfrac{-8+50}{-42} = \dfrac{42}{-42} = -1$

$y = \dfrac{\begin{vmatrix} 8 & 2 \\ 2 & -10 \end{vmatrix}}{-42} = \dfrac{-80-4}{-42} = \dfrac{-84}{-42} = 2$

$\boxed{(-1, 2)}$

Solve a 3 X 3 Sytem Using Cramer's Rule

Cramer's Rule for a 3x3 System

Let A be the coefficient matrix of this linear system:

$$ax + by + cz = j$$
$$dx + ey + fz = k$$
$$gx + hy + iz = l$$

If $\det A \neq 0$, then the system has exactly one solution. The solution is:

$$x = \frac{\begin{vmatrix} j & b & c \\ k & e & f \\ l & h & i \end{vmatrix}}{\det A}, \quad y = \frac{\begin{vmatrix} a & j & c \\ d & k & f \\ g & l & i \end{vmatrix}}{\det A}, \text{ and } \quad z = \frac{\begin{vmatrix} a & b & j \\ d & e & k \\ g & h & l \end{vmatrix}}{\det A}$$

16. Use Cramer's Rule to solve this system: $x - 2y + 3z = -14$; $x - y - z = 0$; $-x + 2y + z = 2$

$|A| = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{vmatrix} \begin{matrix} 1 & -2 \\ 1 & -1 \\ -1 & 2 \end{matrix}$

$= (-1 - 2 + 6) - (3 - 2 - 2)$

$= 3 - (-1) = 4$

$x = \dfrac{\begin{vmatrix} -14 & -2 & 3 \\ 0 & -1 & -1 \\ 2 & 2 & 1 \end{vmatrix}}{4}$

$x = \dfrac{-4}{4} = -1$

$y = \dfrac{\begin{vmatrix} 1 & -14 & 3 \\ 1 & 0 & -1 \\ -1 & 2 & 1 \end{vmatrix}}{4}$

$y = \dfrac{8}{4} = 2$

$-(-1) + 2(2) + z = 2$

$1 + 4 + z = 2$

$z = -3$

$\boxed{(-1, 2, -3)}$

Application: Set up a system and solve using Cramer's Rule.

17. The atomic weights of three compounds are shown. Use a linear system and Cramer's rule to find the atomic weights of carbon (C), hydrogen (H), and oxygen (O).

Compound	Formula	Atomic Weight
Methane	CH_4	16
Glycerol	$C_3H_8O_3$	92
Water	H_2O	18

$C + 4H = 16$

$3C + 8H + 3X = 92$

$2H + X = 18$

$$|A| = \left|\begin{array}{ccc} 1 & 4 & 0 \\ 3 & 8 & 3 \\ 0 & 2 & 1 \end{array}\right| \begin{array}{cc} 1 & 4 \\ 3 & 8 \\ 0 & 2 \end{array}$$

$= (8+0+0) - (0+6+12)$

$8-18 = -10$

$$C = \frac{\left|\begin{array}{ccc} 16 & 4 & 0 \\ 92 & 8 & 3 \\ 18 & 2 & 1 \end{array}\right|}{-10} = \frac{-120}{-10} = 12$$

$12 + 4H = 16$

$4H = 4$

$H = 1$

$2(1) + X = 18$

$X = 16$

$$\left|\begin{array}{ccc} 16 & 4 & 0 \\ 92 & 8 & 3 \\ 18 & 2 & 1 \end{array}\right| \begin{array}{cc} 16 & 4 \\ 92 & 8 \\ 18 & 2 \end{array} = (128 + 216 + 0) - (0 + 96 + 368)$$

$344 - 464$

$= -120$

The atomic weight of carbon is 12, of hydrogen is 1, and oxygen is 16.

Practice: You will need to show work clearly on a separate sheet of paper.

Find the value of each given determinant.

1) $\begin{vmatrix} 3 & 6 \\ 5 & 3 \end{vmatrix} = -21$ **2)** $\begin{vmatrix} 6 & -2 \\ 2 & 1 \end{vmatrix} = 10$ **3)** $\begin{vmatrix} 4 & -2 \\ -8 & 4 \end{vmatrix} = 0$

4) $\begin{vmatrix} 4 & 3 & 1 \\ 0 & 2 & 3 \\ 4 & 2 & 5 \end{vmatrix} = 44$ **5)** $\begin{vmatrix} 5 & 2 & -1 \\ 3 & -2 & 5 \\ 1 & 1 & 3 \end{vmatrix} = -68$ **6)** $\begin{vmatrix} 0 & 3 & 2 \\ -4 & -1 & 2 \\ 5 & 2 & -2 \end{vmatrix} = 0$

7) Find the area of a triangle with the given vertices: (0,0), (0,50), (70,20).

The area is 1750 square units.

Use Cramer's Rule to solve the linear system.

8) $2x + y = 11$, $-3x + 2y = 1$

(3,5)

9) $4x - 5y = 6$, $7x - 12y = 4$

(4,2)

10) $-6x + 4y + z = 32$, $5x + 2y + 3z = 13$, $x - y + z = -5$

(−1,6,2)

Solve for x.

11) $\begin{vmatrix} 2x & 4 \\ x & 3 \end{vmatrix} = 12$

$x = 6$

12) $\begin{vmatrix} 3x & 16 \\ 12 & x \end{vmatrix} = 72$

$x = \pm\sqrt{88}$ or $x = \pm 2\sqrt{22}$

13) $\begin{vmatrix} x & -x \\ 5 & x \end{vmatrix} = -6$

$x = -3, -2$

14) $\begin{vmatrix} 2x & 2x \\ -6 & x \end{vmatrix} = -18$

$x = -3$

Choose 15 OR 16 to solve.

15) $\begin{vmatrix} x & 2 & 4 \\ 9 & 1 & -1 \\ 3 & 2 & x \end{vmatrix} = 26$ $x = 2, 14$

16) $\begin{vmatrix} 2x & 3 & 0 \\ 4 & 2 & -1 \\ 2 & 3 & x \end{vmatrix} = 3x^2 - 9x + 34$ $x = -8, 5$