Accelerated Precalculus
Identity and Inverse Matrices Notes

Name:________________________________

Lesson Opener

Find each product.

1. $\begin{bmatrix} -1 & 5 \\ 3 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1+0 & 0+5 \\ 3+0 & 0+8 \end{bmatrix} = \begin{bmatrix} -1 & 5 \\ 3 & 8 \end{bmatrix}$

$A \; I = A$

$\begin{bmatrix} 4 & 0 & 6 \\ 7 & -5 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$2\times3 ; 3\times3$

2. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 6 \\ 7 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 4+0 & 0+0 & 6+0 \\ 0+7 & 0-5 & 0+4 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 6 \\ 7 & -5 & 4 \end{bmatrix}$

$I \; A = A$

3. $\begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -1 \\ -2 & 3 & -3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2+0+0 & 0-3+0 & 0+0+1 \\ 4+0+0 & 0+2+0 & 0+0-1 \\ -2+0+0 & 0+3+0 & 0+0-3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 2 & -1 \\ -2 & 3 & -3 \end{bmatrix}$

$3\times3 ; 3\times3$

What did you notice about each product?

The product is the same as one of the original matrices.

An identity matrix of multiplication is a __square__ matrix with __1s__ on the main diagonal (upper left to lower right) and __0__ for all other entries.

Note: If A is a square matrix with dimensions n x n and I is the identity matrix with dimensions n x n, then $AI = IA$. $= A$

2 X 2 multiplicative identity:

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 X 3 multiplicative identity:

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1×1

$I = \begin{bmatrix} 1 \end{bmatrix}$

Write the 4 X 4 identity matrix:

$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Find each product.

4. $\begin{bmatrix}-7 & 2\\ 4 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 4 & 7\end{bmatrix} = \begin{bmatrix}-7+8 & -14+14\\ 4-4 & 8-7\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

5. $\begin{bmatrix}3 & 1\\ 6 & 3\end{bmatrix}\begin{bmatrix}1 & -\frac{1}{3}\\ -2 & 1\end{bmatrix} = \begin{bmatrix}3-2 & -1+1\\ 6-6 & -2+3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

What do you notice?

The product is an identity matrix.

inverse of A

If A and A^{-1} are inverse matrices, then $A \cdot A^{-1} = I$ and $A^{-1} \cdot A = I$.

Are the following matrices inverses? Show work and explain.

Are A & B inverse matrices?

6. $\underset{A}{\begin{bmatrix}1 & -\frac{3}{2}\\ -1 & 2\end{bmatrix}}\underset{B}{\begin{bmatrix}4 & 3\\ 2 & 2\end{bmatrix}} = \begin{bmatrix}4-3 & 3-3\\ -4+4 & -3+4\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

Since $AB = I$, then A and B are inverses.

Let $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix}$. The inverse matrix is $A^{-1} = \frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$ where $|A| = ad - bc$ and $|A| \neq 0$.

Find the inverse matrix if it exists. If it does not exist, explain why.

7. $A = \begin{bmatrix}4 & -5\\ -2 & 3\end{bmatrix}$

$|A| = 12 - 10$
$= 2$

$A^{-1} = \frac{1}{2}\begin{bmatrix}3 & 5\\ 2 & 4\end{bmatrix}$

$A^{-1} = \begin{bmatrix}\frac{3}{2} & \frac{5}{2}\\ 1 & 2\end{bmatrix}$

$\begin{bmatrix}\frac{3}{2} & \frac{5}{2}\\ 1 & 2\end{bmatrix}\begin{bmatrix}4 & -5\\ -2 & 3\end{bmatrix}$

$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

8. $B = \begin{bmatrix}8 & -4\\ 5 & -3\end{bmatrix}$

$|B| = -24 + 20$
$= -4$

$B^{-1} = -\frac{1}{4}\begin{bmatrix}-3 & 4\\ -5 & 8\end{bmatrix}$

$\begin{bmatrix}\frac{3}{4} & -1\\ \frac{5}{4} & -2\end{bmatrix}$

9. $C = \begin{bmatrix}-2 & 3\\ -8 & 12\end{bmatrix}$ $|C| = -24 + 24 = 0$

Since $|C| = 0$, C^{-1} does not exist because $\frac{1}{0}$ is undefined.

Find the inverse of a 3 X 3 Matrix using technology.

10. $A=\begin{bmatrix}1 & -1 & 0\\ 1 & 0 & -1\\ 6 & -2 & -3\end{bmatrix}$

11. $B=\begin{bmatrix}2 & -1 & 3\\ 0 & 4 & 2\\ 1 & 6 & 5\end{bmatrix}$

Solving Matrix Equations

Recall: Solve the linear equation $\frac{1}{2}x+4=9$ $\quad -4 \quad -4$ $\quad 2\cdot\frac{1}{2}x=5\cdot 2 \quad 1x=10$

1. Find unknown matrix X such that

$\underset{A}{\begin{bmatrix}4 & 7\\ 1 & 2\end{bmatrix}}\underset{X}{X}=\underset{B}{\begin{bmatrix}9 & 12 & 0\\ -4 & 5 & -2\end{bmatrix}}$

$|A|=8-7=1$

$AX=B$

$A^{-1}AX=A^{-1}B$

$IX=A^{-1}B$

$X=A^{-1}B$

$X=\frac{1}{1}\begin{bmatrix}2 & -7\\ -1 & 4\end{bmatrix}\begin{bmatrix}9 & 12 & 0\\ -4 & 5 & -2\end{bmatrix}=\begin{bmatrix}46 & -11 & 14\\ -25 & 8 & -8\end{bmatrix}$

So what have we found? How can we check the answer?

$\begin{bmatrix}4 & 7\\ 1 & 2\end{bmatrix}\begin{bmatrix}46 & -11 & 14\\ -25 & 8 & -8\end{bmatrix}=\begin{bmatrix}9 & 12 & 0\\ -4 & 5 & -2\end{bmatrix}$

2. Solve the matrix equation $\begin{bmatrix}-4 & 2\\ 8 & 1\end{bmatrix}X=\begin{bmatrix}-16 & 6\\ 22 & 13\end{bmatrix}$

$AX=B \rightarrow X=A^{-1}B$

$|A|=-4-16=-20$

$X=-\frac{1}{20}\underbrace{\begin{bmatrix}1 & -2\\ -8 & -4\end{bmatrix}\begin{bmatrix}-16 & 6\\ 22 & 13\end{bmatrix}}=-\frac{1}{20}\begin{bmatrix}-60 & -20\\ 40 & -100\end{bmatrix}=\begin{bmatrix}3 & 1\\ -2 & 5\end{bmatrix}$

Show all this work!

3. Solve the matrix equation $\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} X - \begin{bmatrix} -1 & 1 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 8 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 5 & 7 \end{bmatrix}$

$+\begin{bmatrix} -1 & 1 \\ 5 & 7 \end{bmatrix}$

$|A| = -8+18 = 10$

$\begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix} X = \begin{bmatrix} 3 & 7 \\ 13 & 9 \end{bmatrix}$ $X = \frac{1}{10}\begin{bmatrix} -2 & 3 \\ -6 & 4 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 13 & 9 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 33 & 13 \\ 34 & -6 \end{bmatrix}$

$X = \begin{bmatrix} \frac{33}{10} & \frac{13}{10} \\ \frac{17}{5} & -\frac{3}{5} \end{bmatrix}$

$\frac{34}{10}$ $-\frac{6}{10}$

A system of linear equations can be rewritten as a matrix equation.

$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$ can be written as $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$

coefficient matrix → ; **variable matrix** ; ← **constant matrix**

$AX = B$

Rewrite the system of equations as a matrix equation and use inverse matrices to solve for x and y.

4. $\begin{cases} 2x + 3y = 15 \\ x - 2y = -17 \end{cases}$ $(-3, 7)$

$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 15 \\ -17 \end{bmatrix}$ (A, X, B)

$|A| = -4-3 = -7$

$\boxed{X} = -\frac{1}{7}\begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 15 \\ -17 \end{bmatrix}$

$= -\frac{1}{7}\begin{bmatrix} 21 \\ -49 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \end{bmatrix} \to \boxed{(-3, 7)}$

5. $\begin{cases} 4x + 2y = -10 \\ x - 3y = 15 \end{cases}$ $(0, -5)$

$\begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -10 \\ 15 \end{bmatrix}$

$|A| = -14$

$X = -\frac{1}{14}\begin{bmatrix} -3 & -2 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} -10 \\ 15 \end{bmatrix}$

$= -\frac{1}{14}\begin{bmatrix} 0 \\ 70 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix} \to \boxed{(0, -5)}$

Solve a 3 X 3 system using inverse matrices and technology.

6. $\begin{aligned} 2x + 3y + 4z &= 7 \\ -x + 5y + 2z &= 6 \\ -3x + 6y &= 3 \end{aligned}$